Half Way Around the World

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Ethnomusicology / Ethnomathematics

- How did the Samba come from Africa to Brazil?

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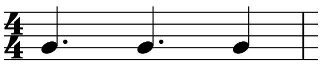
- Why is Calypso $=$ Baião $=$ 

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- Activity: Analyse the symmetries of the carpet.

Modular arithmetics

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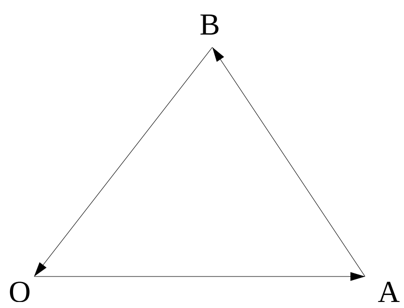
- Cyclic symmetry: 

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- Applications: Algebraic Geometry, Physics, Combinatorics, Coding Theory (Sloane 1977), Any more???

Vector spaces and Groups

Let $V \cong \mathbf{C}^n$ be a complex vector space having a basis $\mathcal{B} = (\mathbf{e}_1, \ldots, \mathbf{e}_n)$ and let $\mathbf{x} = (x_1, \ldots, x_n)$ be a basis of the algebraic dual V^* satisfying $\forall 1 \leq i, j \leq n : x_i(\mathbf{e}_j) = \delta_{ij}$. Let $G \leq \mathrm{GL}\,(V)$ be a finite group acting linear on V from the left, i. e. for all $g \in G$ the mapping $V \to V, \mathbf{v} \mapsto g.\mathbf{v}$ is linear and bijective. In coordinates this can be expressed as $[g.\mathbf{v}]_{\mathcal{B}} = [g]_{\mathcal{B},\mathcal{B}}[\mathbf{v}]_{\mathcal{B}}$. In other words, we may view G as a matrix group.

Symmetric Algebra

The elements of V^* are linear transformarions, in particular $x_1, \ldots, x_n \in V^*$. Viewing $x_1, \ldots, x_n$ as variables, we can view $S(V^*)$, the *Symmetric Algebran* on V^*, as the algebra $R := \mathbf{C}[\mathbf{x}] := \mathbf{C}[x_1, \ldots, x_n]$ of polynomial functions $V \to \mathbf{C}$ and as polynomials in these variables, linear forms in case of degree one. It is naturally graded as

$$R = \bigoplus_{d \in \mathbf{N}} R_d,$$

where R_d denotes the vector space generated by all polynomials of total degree d. In particular, $R_0 = \mathbf{C}$ and $R_1 = V^*$.

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- $R^G = \bigoplus_{d \in \mathbf{N}} R_d^G$, where $R_d^G = \{ f \in R_d : \forall g \in G : g.f = f \}$, we seek to find the dimesions of the summands.

Intermezzo

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- Simultaneously, email from LP arrived, approved!